Conformalized Mean Curvature Flow

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Barrett Lectures, UTK

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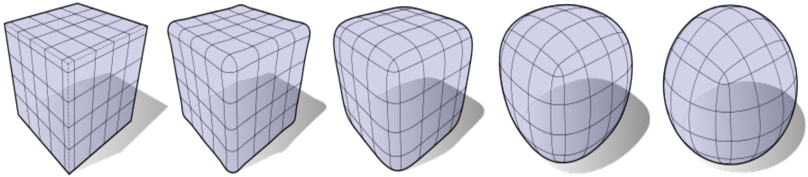
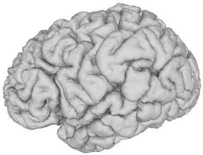


Shape Comparison

- Complex shapes in nature: human brains, proteins, bones, etc.
- Application: medicine, anthropology, image processing, etc.
- Goal: Compare any two different compact genus-zero surfaces without boundary (i.e. 2-sphere $\mathbb{S}^2$)



Shape Comparison in Biology

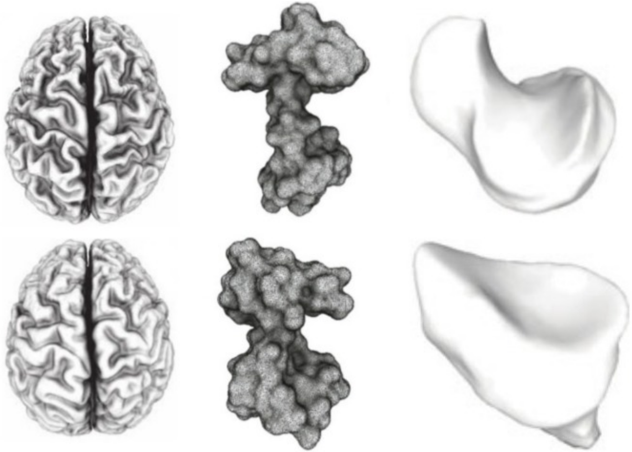


Figure: Source: Hass & Koehl

One possible way: Conformal Maps diagram from Hass & Koehl

A framework proposed by Hass and Koehl: comparison of surfaces

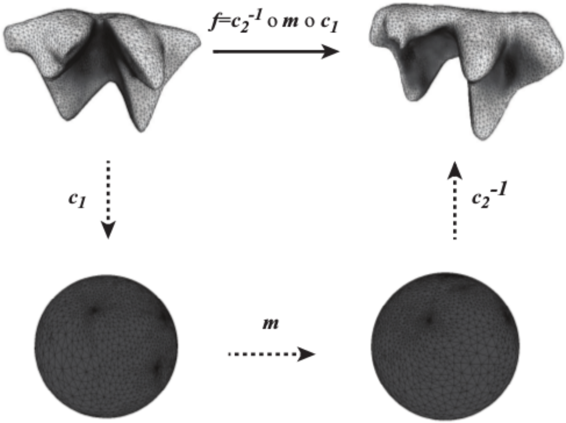


Figure: Hass, Koehl “Comparing shapes of genus-zero surfaces" **2017**

Why Conformal Mapping?

Possible actions on tangent plane

smooth	$\supset$ **conformal** $\supset$	isometric
translate	*translate*	*translate*
rotate	*rotate*	*rotate*
scale	*scale*	
shear		

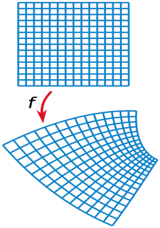


Figure: Wikipedia

- Given Riemann surfaces M and two metrics $g, \tilde{g}$ on M. We say $\tilde{g}, g$ are conformally equivalent if $\exists$ positive $\rho \in C^\infty(M)$ such that $\tilde{g} = \rho g$.
- Given two surfaces (M, g), (N, h). $f : M \to N$ is a conformal map if $\exists$ positive $\tau \in C^\infty(M)$ such that the pullback metric $f^*(h) = \tau g$.

Poincaré-Klein-Koebe Uniformization theorem

Theorem

*A closed **genus-zero** Riemann surface M is conformally equivalent to the Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Restatement:

Given closed genus-zero Riemann surface (M, g)*, there exists a metric* $\tilde{g}$ *conformal to* g *and* $\tilde{g}$ *is of **constant Gaussian curvature*** 1.

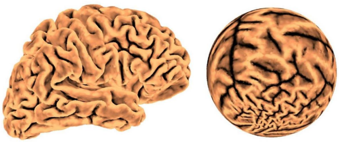


Figure: By Gary Choi (Harvard) on mathworks

WHY on a sphere?

(i). Canonical domain for data comparison

(ii). 6 deg. freedom left: Möbius transform $PSL(2, \mathbb{C})$

Discrete Conformal Mapping

- Surface representation - a triplet $\mathcal{M} := (\underbrace{V}_{\text{vertices}}, \underbrace{E}_{\textit{edges}}, \underbrace{T}_{\textit{faces}})$
- Discrete metric $\ell\colon E \to \mathbb{R}_{>0}$ assigns positive value to edges, i.e. $\ell(e_{ij}) = \ell_{ij}$ such that all triangle inequalities holds for $t_{ijk} \in T$.
- ℓ and $\tilde{\ell}$ on M are (*discrete*) conformally equivalent if $\exists\, u\colon V \to \mathbb{R}$ such that for $v_i, v_j \in V$, $\tilde{\ell}_{ij} = e^{\frac{u(v_i)+u(v_j)}{2}} \ell_{ij}$.

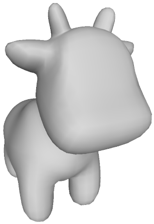


Figure: A smooth Spot

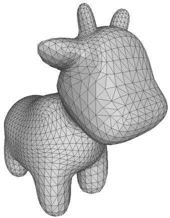


Figure: A discretized Spot

Discrete Conformal Mapping: framework revisited

Setting

- *Given discrete surface $M = (V, E, T)$ in $\mathbb{R}^3$.*
- *Use induced metric ℓ from $\mathbb{R}^3$.*
- *Want an algorithm to give map $\boldsymbol{c}: V \to \mathbb{R}^3$ with output (V', E', T'), V' lie on $\mathbb{S}^2 \subset \mathbb{R}^3$.*
- *Want that new induced metric $\tilde{\ell}$ is conformal to initial ℓ.*
- *Uniformization Theorem: Such (continuous) conformal map exists!*

Figure: *Y. Wang (ASU)*

Problem

*A **robust** discrete conformal mapping algorithm applicable on a wide range of shapes **doesn't exist yet**!*

Already tried - Discrete Ricci Flow and Bobenko's Method

Intrinsic geometric flow is an evolution of Riemannian metric

- Discrete Ricci Flow: Distribute total curvature (4π) evenly.

  Chow, Luo "Combinatorial Ricci flows on surfaces" **2003**
  Jin, Kim, Luo, Gu "Discrete Surface Ricci Flow" **2008**

  - Drawback: Restrictive, Mesh degeneracy

- Bobenko's Method: Minimize convex energy functional

  Springborn, Schöder, Pinkall "Conformal Equivalence of Triangle Meshes" **2008**
  Bobenko, Pinkall, Springborn "Discrete conformal maps and ideal hyperbolic polyhedra" **2010**

  - Drawback: Triangle inequalities might fail

Mean Curvature Flow (MCF)

Let $\Phi_t : M \to \mathbb{R}^3$ be a smooth family of immersions and $g_t(\cdot,\cdot)$ be the metric induced by Φ_t at time t. Φ_t is a solution to the MCF if

$$\frac{\partial \Phi_t}{\partial t} = \Delta_{g_t} \Phi_t (= -2 H_t \hat{N}_t) \tag{1}$$

$H_t(p)$:scalar mean curvature, $\hat{N}_t(p)$: **outward** unit surface normal.
Δ_{g_t}: Laplace-Beltrami operator defined w.r.t. g_t.

Singularities form when surface collapse at a point ($\kappa(p) \to \infty$).
Hence MCF is not conformal in nature.

Figure: Alsing, Paul M. *et al.* “Simplicial Ricci Flow" **2014**

Strategy: From MCF to Conformal Map

Kazhdan, Solomon, Ben-Chen “Can Mean-Curvature Flow be Modified to be Non-singular?" 2012:

1. Apply finite-elements discretization to MCF
2. Identify numerical instabilities
3. Propose modified flow that resolves instabilities
4. Convergence of the new flow?

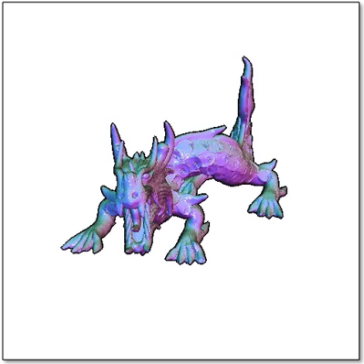


Figure: M. Kazhdan, J. Solomon, M. Ben-Chen

1. Finite-elements Method (FEM)

Express immersion

$$\Phi_t(p) = \sum_{i=1}^{N} x_i(t) B_i(p)$$

where $\{B_1, B_2, \cdots B_N\} : M \to \mathbb{R}$ is a set of function basis and a set of coefficient vectors $X(t) = \{x_1(t), x_2(t), \cdots, x_N(t)\} \subset \mathbb{R}^3$.

Warning: possible $\Phi_t \notin \text{span}\{B_1, \cdots B_N\}$! Using Galerkin formulation:

$$\int_M \left(\frac{\partial \Phi_t}{\partial t} \cdot B_j \right) dA_t = \int_M \left(\Delta_t \Phi_t \cdot B_j \right) dA_t \quad \forall 1 \leqslant i \leqslant N$$

Apply Backward Euler method to discretize $\frac{\partial x_i}{\partial t} \approx \frac{x_i(t+\tau) - x_i(t)}{\tau}$.

1. Finite-elements Method (Method)

When all the dust settles...

$$\left(D^t - \tau L^t\right) X(t+\tau) = D^t X(t) \tag{2}$$

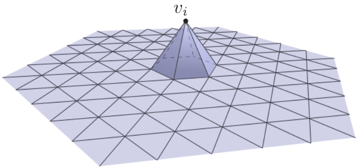
where $D^t := [D^t_{ij}], L^t := [L^t_{ij}]$ are $N \times N$ matrices.

$$D^t_{ij} := \int_M B_i B_j \, dA_t, \quad L^t_{ij} := -\int_M g_t(\nabla_t B_i, \nabla_t B_j) \, dA_t$$

We solve a linear system $Ax = b$ relating $X(t+\tau)$ to $X(t)$.
Use “Hat Basis", piecewise linear function basis $B_i : V \to \mathbb{R}$

$$B_i(v_j) = \delta_{ij}$$ Delta functional

$$\Rightarrow \Phi_t(v_j) = \sum_i x_i(t)\delta_{ij} = x_j(t)$$



An Excursion into Differential Geometry

Linear map $\Lambda : T\Phi_0(p) \mapsto T\Phi_1(p)$, maps orthogonal vectors $\partial_{v_1}, \partial_{v_2} \in T\Phi_0(p)$ to two corresponding vectors, $\partial_{w_1}, \partial_{w_2} \in T\Phi_1(p)$ with $\partial_{w_1} \perp \partial_{w_2}$, i.e.

$$\Lambda := d\Phi_1 \circ d\Phi_0^{-1} \quad \Rightarrow \quad \Lambda\partial_{v_i} = \lambda_i \partial_{w_i}, \quad i = 1, 2 \tag{3}$$

where stretch directions ∂_{w_i}, stretch factors λ_i are time dependent.

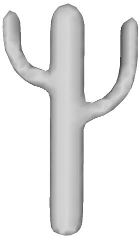


Figure: cactus: initial

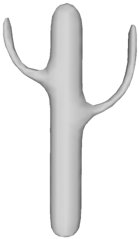


Figure: cactus: MCF 10 steps

2. Numerical Instabilities

Express D^t, L^t using $dA_t = \sqrt{|g_t|\,|g_0|^{-1}}\,dA_0 = \lambda_1\lambda_2\,dA_0$:

$$D^t_{ij} = \int_M B_i \cdot B_j \left(\lambda_1\lambda_2\right) dA_0 \tag{4}$$

$$L^t_{ij} = -\int_M \left(\frac{\lambda_2}{\lambda_1}\frac{\partial B_i}{\partial v_1}\frac{\partial B_j}{\partial v_1} + \frac{\lambda_1}{\lambda_2}\frac{\partial B_i}{\partial v_2}\frac{\partial B_j}{\partial v_2}\right)\, dA_0 \tag{5}$$

Where's Instability? Anisotropic stretching (different magnitudes of stretching along v_1, v_2) $\Rightarrow$ either $\frac{\lambda_2}{\lambda_1}$ or $\frac{\lambda_1}{\lambda_2}$ escapes to infinity

To conclude, L^t might blow up when singularities form!

3. Conformalized Mean Curvature Flow (cMCF)

Idea: replace metric g_t by the closest metric that is conformal to g_0.
Let $\tilde{g}_t$ be "conformalized" metric and $\tilde{\lambda}_1, \tilde{\lambda}_2$ new stretch factors.

$$\tilde{g}_t := \sqrt{|g_t| \ |g_0|^{-1}} g_0 \tag{6}$$

- $\tilde{g}_t$ is conformal to g_0 and $|\tilde{g}_t| = |g_t|$
- $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \sqrt{\lambda_1 \lambda_2} \Rightarrow \tilde{\lambda}_1 \tilde{\lambda}_2 = \lambda_1 \lambda_2$
- $\tilde{D}^t_{ij} = D^t_{ij}$ but $\tilde{L}^t = L^0$ and hence $\tilde{L}^t$ is independent of time.

Figure: Gargoyle under cMCF

3. Conformalized Mean Curvature Flow (cMCF)

Definition

Given $\mathcal{M}, g_0$. Let $\Phi_t : M \to \mathbb{R}^3$ be a smooth family of immersions and $g_t(\cdot,\cdot)$ be the induced metric. Φ_t is a solution to cMCF if:

$$\frac{\partial \Phi_t}{\partial t} = \sqrt{|g_t|^{-1}\ |g_0|}\Delta_{g_0}\Phi_t \tag{7}$$

- Laplace-Beltrami operator stays the same.
- If Φ_t is conformal with respect to g_0, then $\Delta_{g_t} = \sqrt{|g_t|^{-1}\ |g_0|}\Delta_{g_0}$. Recover traditional MCF from cMCF!
- Discretizing eq. (7) gives a varying D^t but a fixed L^t.

cMCF Algorithm by Prof. Michael Kazhdan available at:

`www.cs.jhu.edu/~misha/Code/ConformalizedMCF/Version2/`

3. cMCF: Numerical Results

Spot is back! Step size 0.0001

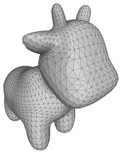


Figure: Spot: Initial

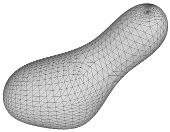


Figure: cMCF - 10 steps

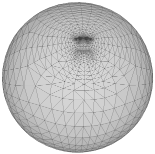


Figure: cMCF - 500 steps

Spot brings a new friend - dinosaur $|V| = 9,794$, $|T| = 19,584$......

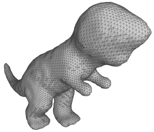


Figure: Dinosaur: Initial

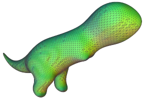


Figure: cMCF - 10 steps

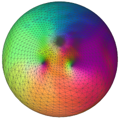


Figure: cMCF - 2000 steps

3. cMCF: Numerical Results

Sophisticated Mesh - Armadillo!!! $|V| = 172,974, |T| = 345,944$

Figure: Armadillo: Initial

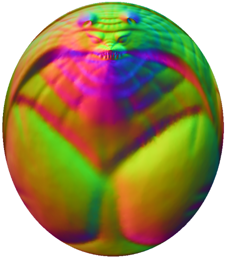


Figure: cMCF - 500 steps

	Initial	500 steps
Area	38129	93009
Volume	237977	2.63748e+06
Sphericity	0.487063	0.992549

3. cMCF: Numerical Results

Back to application - Human Brain $|V| = 65,538, |T| = 131,072$

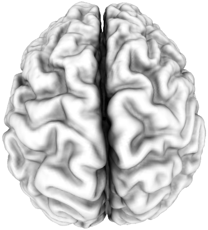


Figure: Brain: Initial

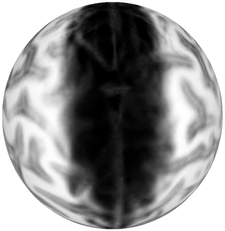


Figure: cMCF - 500 steps

4. cMCF: Convergence

Open question in case of genus-zero surfaces: Φ_t converges under cMCF? Existence of a counter-example?

Kazhdan, Solomon, and Ben-Chen proved:

Proposition

If cMCF converges, i.e. $\Phi_t \xrightarrow[t\to\infty]{} \Phi_\infty$,
then Φ_∞ *is a map onto the sphere if and only if* Φ_∞ *is conformal*

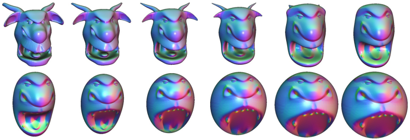


Figure: Source: M. Kazhdan, J. Solomon, M. Ben-Chen

Math journey continues...

- Surfaces with higher genus? Known: limit map is not conformal.
    - No embeddings of closed surfaces with higher genus that are uniformly scaled by traditional MCF.

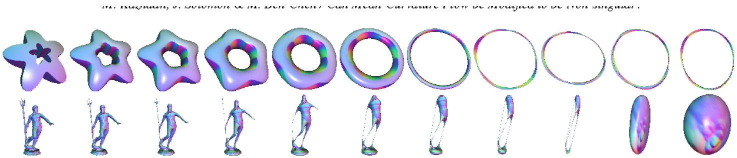


Figure: Source: M. Kazhdan, J. Solomon, M. Ben-Chen

- Surfaces with boundary?
- Use defining function to reduce MCF into one equation. An equivalent formulation of cMCF?

CS journey also continues...

- Existing algorithm not applicable on “raw" meshes from real brain scanning.
- My own implementation of Kazhdan’s algorithm in OpenMesh. Parallel Computing?

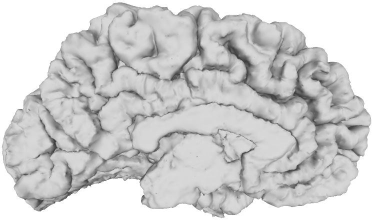


Figure: Half Brain: View 1

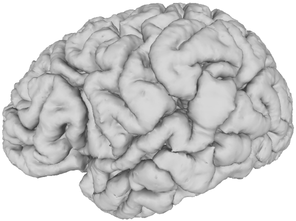


Figure: Half Brain: View 2

An amazing team at UC Davis

Figure: Prof. Joel Hass, source: IAS

Figure: Prof. Patrice Koehl, source: UCD

Figure: Yanwen Luo, source: UCD

Figure: Karry Wong

Thank You!